JOURNAL OF ALGEBRA, 3, 99-114 (1966)

Finite Groups Admitting a Fixed-Point-Free Automorphism of Order $2p$*

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Communicated by Walter Feit

Received February 13, 1965

It is an open question whether a finite group G admitting a fixed-point-free automorphism f is solvable. Thompson [7] has proven that G is nilpotent if f has prime order. Gorenstein and Herstein [5] have proven that G is metanilpotent if f has order 4. We generalize this latter result and prove:

THEOREM 1. *Let G be a finite group admitting a fixed-point-free automorphism f of order $2p$ for a prime p. Then G is solvable if one of the following conditions holds*:

(I) $C_G(f^p)$ *is a 2-group.*

(II) $C_G(f^p)$ *contains a Sylow 2-subgroup of G.*

The proof of (I) is almost trival, since finite groups which are the product of two nilpotent subgroups are solvable, see Kegel [6]. The proof of (II) is based on

THEOREM 2. *If G is a finite group generated by a class D of conjugate involutions such that different elements of D never commute and the centralizer of an element of D is 2-closed and 2′-closed then G is solvable.*

Theorem 2 implies that finite groups of odd order are solvable (Remark 2.2); but we make use of the theorem of Feit and Thompson [3] and the theorem of Brauer and Suzuki [2].

Because of (II), we always can find "large" proper subgroups of non-solvable groups admitting fixed-point-free automorphisms of order $2p$. For the proof of (II) we need informations about the Sylow 2-subgroups of solvable groups admitting fixed-point-free automorphisms of order $2p$ (Lemma 3.4).

In the proof of Theorem 1 we consider a normal complex of involutions D

* Research supported in part by the European Research Office, U. S. Department of the Army.

generating a group H such that different maximal subsets of D generating a 2-group have no element of D in common. If a group H with this property is solvable in general groups admitting a fixed-point-free automorphism of order $2p$ would be solvable. Theorem 2 gives a solution of this question in a special case. We treat two other special cases:

THEOREM 3. *Let G be a finite group generated by a class of conjugate involutions D such that each element of D normalizes exactly one Sylow p-subgroup of G for all primes p. Then G' is a nilpotent group of odd order.*

THEOREM 4. *Let N be a proper normal subgroup of the finite group G and d a 2-element of G such that*

(a) $G = N\{d\}$ *and* $N \cap \{d\} = 1$;

(b) *if S and S^* are different Sylow 2-subgroups of G then* $S \cap S^* \subseteq N$.

Then N is 2-closed.

NOTATIONS

$\lvert M \rvert$	number of elements in a set M;
$\{x; ...\}$	subgroup generated by all x...;
$[x; ...]$	set of all x with ...;
$[G : U]$	index of U in G;
G'	commutator subgroup of G;
$Z(G)$	center of G;
$N_M(T)$	normalizer of T in M;
$C_M(T)$	centralizer of T in M;
$G\{f\}$	(split) extension of G in the holomorph of G with the cyclic group generated by the automorphism f of G;
σ-group (σ-element) :	group (element) such that the primes dividing the order are contained in the set σ;
σ'	set of primes not in σ;
σ-closed	groups in which the product of σ-elements is a σ-element;
Hall σ-subgroup :	σ-group containing for each prime $p \in \sigma$ a Sylow p-subgroup of G;
$g^x = x^{-1}gx = g \circ x$.	

1. A CHARACTERISTIC SUBGROUP OF SOLVABLE GROUPS

Let G be a finite solvable group and let p be a prime. Then we define

$$\rho_p(G) = \{S; S \text{ is a Sylow } p\text{-subgroup of } G'\}.$$

LEMMA 1.1. *Let G be a finite solvable group and let p be a prime. Assume that G is generated by a class of conjugate elements D of order p. Then the following statements hold:*

(a) *If M is a normal subgroup of G such that different elements of DM/M do not commute, then M contains $\rho_p(G)$.*

(b) *Different elements of $D\rho_p(G)/\rho_p(G)$ do not commute.*

(c) *Let T be a maximal subset of D generating a p-group; then $\rho_p(G)$ is the normal closure of $\{T\} \cap G'$ in G'.*

Proof. Since $G'/\rho_p(G)$ is a p'-group and G is generated by a class of conjugate p-elements, (b) is true.

Let G be a group of minimal order such that (a) is false. Since G is solvable, G contains a minimal normal subgroup M which is abelian. We have to show that G'/M is a p'-group. Let N be a normal subgroup of G containing M such that N/M is a minimal normal subgroup of G/M. Then the minimality of G implies that $(G/N)'$ is a p'-group. Since different elements of DM/M do commute, N/M is contained in $Z(G/M)$. Since G'/N is a p'-group, we get

$$G'/M = N/M \otimes H/N$$

for the subgroup H of G' containing M and the p'-elements of G'. Hence H is normal in G. Since DM/M is a class of conjugate elements, we get

$$DM/M = d^H M/M \subseteq \{dH/M\}.$$

This contradiction proves (a).

Let T be a maximal subset of elements of D generating a p-group. Let M be the normal closure of $\{T\} \cap G'$ in G'. Let d and e be elements of D such that $\{e, d\}M/M$ is a p-group. Then

$$\{e, d\}M/M \simeq \{e, d\}/M \cap \{e, d\}.$$

Now, the definition of M implies that $\{e, d\}M/M$ has order p, and hence

$$dM/M = eM/M.$$

Therefore different elements of DM/M do not commute. Hence (b) implies

$$M \supseteq \rho_p(G).$$

Since M is contained in $\rho_p(G)$ the proof of Lemma 1.1 is complete.

2. Proof of Theorem 2

We assume in this section that G is a finite group with the properties

(i) *G is generated by a normal complex D of conjugate involutions*;

(ii) *different elements of D do not commute.*

Remark 2.1. If G satisfies (i) then condition (ii) is equivalent with

(ii′) *the product of (different) elements of D has odd order.*

Proof. We assume that (ii) is true. Let a and b be different elements of D. Then $\{a, b\}$ is a dihedral group. If a and b are not conjugate in $\{a, b\}$ there is an involution z in $Z(\{a, b\})$ that is different from a and b. Then b and $za \neq a$ are conjugate in $\{a, b\}$. Hence za belongs to D by (i). But a and az commute, which is impossible because of (ii).

Of course (ii′) implies (ii).

(2.A) *D is a class of conjugate elements of G. If M is a normal subgroup of G with $| MD/M | \geqslant 2$ and if U is a subgroup of G with $| U \cap D | \geqslant 2$ then* (i) *and* (ii) *are true for G/M and $\{U \cap D\}$.*

Proof. (2.A) is a trivial consequence of Remark 2.1.

(2.B) *If d is an element of D then $C_G(d)$ contains each Sylow 2-subgroup of G containing d.*

Proof. Let S be a Sylow 2-subgroup of G containing d. Then (2.A) implies $| S \cap D | = 1$ and d is contained in $Z(S)$.

Definition. *Let i be an automorphism of G and assume $i^2 = 1$. Then D_i is the set of elements of D centralized by i. Let a be an element of D_i. Then we define*

$$A_{i,a} = [x \in D; \{ixi, x\} \cap D_i = a],$$

where ixi is the image of x under transformation by i.

(2.C) *Let i be an automorphism of G with $i^2 = 1$ and let a be an element of D_i. Then the following statements holds*:

(a) $D = A_{i,a} \circ D_i$;

(b) $| D | = | A_{i,a} || D_i |$;

(c) $A_{i,a} = \{A_{i,a}\} \cap D$;

(d) $D_i = A_{ia,a}$.

Proof. We gave a proof of (2.C) in a similar situation in [*3*, proof of Satz 4.1, steps (7), (10), (11)]. Therefore we indicate only some points of the proof.

If x is an element of D it is contained in exactly one $A_{i,a(x)}$: This is trivial if x is contained in D_i ; if x is not contained in D_i the group $\{x, ixi\}$ is a dihedral group and i fixes exactly one element of $\{x, ixi\} \cap D$, since (2.B) implies that the Sylow 2-subgroups of $\{x, i\}$ have either order 2 or 4. Because of (2.A), D_i is a class of conjugate elements in $\{D_i\}$. Hence we can show

$$w^{-1}A_{i,a}w = A_{i,w^{-1}aw} \qquad \text{for} \qquad w \in \{D_i\}.$$

Hence (a) and (b) are true. Since ia fixes the element a and no other element of D_i , we get (d). Now, (d) and (2.A) imply (c).

(2.D) *If Q is a Sylow q-subgroup of $C_G(d)$ for an element d of D then Q is contained in a Sylow q-subgroup P of G which is normalized by d.*

Proof. We assume that G has minimal order such that (2.D) is false. If G is solvable we get a contradiction from Lemma 1.1 and the theorems of Hall. If G' has odd order (2.D) is trivial because of the Frattini argument. Hence G' has even order. The lemma of Burnside [*8*, p. 169] and the theorem of Brauer and Suzuki [*2*] imply that there is an involution t in G' which is not contained in D. Since G is minimal, (2.C) implies that D_t contains more than one element. Let P be a maximal q-subgroup of G such that d normalizes P and Q is contained in P. Because of (2.C), either $C_G(d)$ or $\{A_{t,a}\}$ or $\{A_{ta,a}\}$ for an element a of D_t contains a q-subgroup which is different from 1. Hence (2.C) and the minimality of G imply that P contains more than one element. Because of (2.A), the Frattini-argument shows

$$N_G(P) = \{N_D(P)\}C_{N_G(P)}(a) \qquad \text{for} \qquad a \in N_D(P).$$

Let R be a Sylow q-subgroup of $C_{N_G(P)}(a)$ containing $C_P(a)$. The latter equation shows that PR is a Sylow q-subgroup of $N_G(P)$ normalized by a. Since $C_P(a)$ is a Sylow q-subgroup of $C_G(a)$, the maximality of P implies $P = PR$. Hence P is a Sylow q-subgroup of G. This contradiction proves (2.D).

(2.E) *Let t be an automorphism of G such that $t^2 = 1 \neq t$ and $C_{G'}(t) \supseteq C_{G'}(d)$ for an element d of $C_D(t)$. Then $\{D_t\}'$ and $\{D_{dt}\}'$ are normal subgroups of G.*

Proof. We can assume that d and t are different elements of $H = G\{t\}$. Then $C_G(dt)$ contains $C_G(d)$. Set $w = t$ or dt and $v = dt$ or t. The hypothesis implies together with (2.C)

$$| A_{w,d} | = [C_G(v) : C_G(d)]$$

and

$$\begin{aligned} | D | &= [G : C_G(d)] = [G : C_G(w)][C_G(w) : C_G(d)] \\ &= | A_{w,d} || A_{v,d} | = [C_G(v) : C_G(d)][C_G(w) : C_G(d)]. \end{aligned}$$

Hence

$$| w^H | = | w^G | = [C_G(v) : C_G(d)] = | A_{w,d} |.$$

Now, (2.C) implies

$$\{w^H\}' = \{A_{d,w}\}'.$$

Since $\{w^H\}'$ is normal in H, we get (2.E).

(2.F) *Assume that $C_G(d)$ for an element d of D is 2-closed and $2'$-closed. Then G is solvable.*

We assume that G is a group of minimal order such that (2.F) is false.

(1) *$G' = G''$ is a minimal normal subgroup of G.*

Proof. Let M be a minimal normal subgroup of G contained in G'. Then $C_M(d)$ is 2-closed and $2'$-closed. If (1) is false we get that M is (contained in $Z(G)$ or) solvable. (2.A) and (2.B) imply for each element x of D

$$x^M = M\{x\} \cap D.$$

Hence

$$| D | = | x^M || DM/M | \qquad \text{for} \qquad x \in D.$$

Another application of (2.A) shows

$$| C_{G/M}(dM/M)| = | C_G(d)M/M |.$$

Hence $C_{G/M}(dM/M)$ is 2-closed and $2'$-closed. The minimality of G implies that G/M is solvable. This contradiction proves (1).

(2) *If s is an involution in G and p a prime then s normalizes a Sylow p-subgroup of G which contains a Sylow p-subgroup of $C_G(s)$.*

Proof. We assume that (2) is false for an involution s and a prime p. Then $p \neq 2$ and s is not contained in D by (2.D). Since (2.D) implies that an element d of D_s normalizes a Sylow p-subgroup Q^* of $\{D_s\}$, we get from (2.A)

$$C_G(s) = \{D_s\}C_G(\{s, d\}).$$

Hence d normalizes a Sylow p-subgroup Q_0 of $N_{C_G(s)}(Q^*)$ that is a Sylow p-subgroup of $C_G(s)$. Hence

$$sQ_0s = Q_0 = dQ_0d.$$

Let Q be a maximal p-subgroup of G such that

$$sQs = Q = dQd \supseteq Q_0.$$

Because of (2.D), there is a Sylow p-subgroup P of G normalized by d.

We assume that Q is trivial. Then p does not divide $|C_G(s)|$. The hypothesis implies $C_P(d) = 1$, since s centralizes at least one Sylow p-subgroup of $C_G(d)$. Therefore (2.C) implies that $\{D_{sd}\}$ contains a Sylow p-subgroup R of G since $\{D_s\}$ is a p'-group. Because of (2.D), R is normalized by d and s. Hence Q is not trivial. (2.A) implies

$$\{d, s\} \subseteq N_G(Q) = \{N_D(Q)\}C_{N_G(Q)}(d).$$

Since Q is not trival, we get from (1) and the minimality of G that $\{N_D(Q)\}$ is solvable. The theorem of Feit and Thompson implies that $C_G(d)$ is solvable, since it is 2-closed. Hence $N_G(Q)$ is solvable. Since $\{s, d\}$ is a 2-group, the theorems of Hall imply that there is a Hall $[2, p]$-subgroup H of $N_G(Q)$ containing s, d, and Q. Now, (2.A) implies

$$H = \{d^H\}C_H(d).$$

Because of Lemma 1.1, $\{d^H\}'$ is a p-group. If R_0 is a Sylow p-subgroup of $C_H(d)$ then $R = \{d^H\}'R_0$ is a Sylow p-subgroup of H normalized by d. The hypothesis implies that $C_H(d)$ is nilpotent. Hence R is normalized by s. Since Q is not a Sylow p-subgroup of G, we get

$$Q \subset R = sRs = dRd.$$

Hence Q is not maximal.

(3) *Let s be an involution of G which is not contained in D. Then D_s is contained in exactly one maximal subset $N(s)$ of D generating a proper subgroup of G.*

Proof. Let d ba an element of D_s and let M be a maximal subset of D containing D_s and generating a proper subgroup of G. Let m be an element of M. Because of (2.C), m is contained in exactly one set $A_{s,a(m)}$ for $a(m) \in D_s$. Since M is maximal we get from (2.A)

$$M = \{M\} \cap D.$$

Therefore $\{m, a(m)\} \cap D$ is contained in M and $M = sMs$. Hence (2.E) implies

$$M \subseteq N_G(\{A_{sd,d}\}') = N_G(\{D_s\}').$$

Hence (3) is a consequence of (1) and the maximality of M.

(4) *Let $s \neq d$ be an involution in $C_G(d)$ for $d \in D$. Then $C_G(\{d, s\})$ contains exactly the involutions d, s, ds.*

Proof. Let t be an involution of $C_G(\{s, d\})$ which is not contained in

$\{d, s\}$. If D_t contains D_s we get $D_t \supset D_s$ since st is not contained in $Z(G)$. Hence $W = D_t \cap D_{sd}$ contains more than one element. Because of

$$W = (A_{s,d})_t = (A_{s,d} \cap D_t)_t ,$$

we get from (2.E) that $\{W\}'$ is normal in $\{D_t\}$ and in $\{A_{s,d}\}$, which is impossible.

Therefore we can assume

$$D_t \cap D_{sd} = D_t \cap A_{s,d} \neq d \neq D_t \cap D_s \subset D_s .$$

Hence $\{D_t \cap D_s\}'$ is normal in $\{D_t\}$ and in $\{D_s\}$ by (2.F). From (3) and (1) we get $N(s) = N(t)$. But $\{D_t \cap D_{sd}\}'$ is normal in $\{D_t\}$ and $\{D_{sd}\}$. Hence

$$D_s \subseteq N(s) = N(t) = N(sd) \supseteq D_{sd}.$$

Now, (2.C) implies $N(s) = D$, which is impossible.

(5) *If S is a Sylow 2-subgroup of G then $Z(S)$ contains exactly one involution* $(Z(S) \cap D)$.

Proof. If $Z(S)$ contains an involution $s \neq Z(S) \cap D = d$ then (4) implies that S contains exactly three involutions. Hence different elements of s^G and different elements of $(sd)^G$ do not commute. Therefore (2.E) implies that $\{s^G\}$ and $\{(sd)^G\}$ are proper subgroups of G. Hence (1) implies

$$G' = \{(ds)^G\} = \{s^G\} \supseteq \{d^G\} = G,$$

which is impossible.

Now, we can finish the proof. Let S be a Sylow 2-subgroup of G. Then S contains an involution s different from $D \cap Z(S)$ because of (1), the theorem of Brauer and Suzuki, and the theorem Feit and Thompson. Then (5) implies that s^S contains an involution $t \neq s$ that commutes with s. Hence (4) implies

$$s = dt \qquad \text{for} \qquad d = Z(S) \cap D.$$

Let M be a minimal normal subgroup of $\{D_s\}$. Then M has odd order by Lemma 1.1 and the minimality of G. Let w be an element of S with $t = s^w$. Set

$$C = C_M(d), \qquad I = M \cap \{d^M\}, \qquad J = I^w.$$

Since M is an elementary abelian p-group of odd order, the hypothesis implies

$$M = C \otimes I \qquad \text{and} \qquad M^w = C \otimes J.$$

Because of (2), G contains a Sylow p-subgroup P such that

$$sPs = dPd = P$$

and $C_P(s)$ is Sylow p-subgroup of $C_G(s)$. Hence (2.C) implies

$$P\{d\} = C_P(d)\{d^P\} = \{(d^P)_s, (d^P)_t\}C_P(d)$$

Therefore $N = \{M, J\}$ is *a* p-subgroup of G. Since M is normal in $\{D_s\}$ and M^w is normal in $\{D_t\}$ we get from (2.E)

$$N = ICJ = M^w M,$$

and N is elementary abelian since M is elementary abelian. Hence (3) and (2.E) imply that I is normal in $\{D_t, I\}$ and J is normal in $\{D_s, J\}$. Hence N is a normal subgroup of $G = \{D_t, D_s\}$, which is impossible.

Remark 2.2. Let E be a finite simple group of odd order. There is an automorphism d of $E \otimes E$ such that E is not normalized by d and $d^2 = 1$. If $E' = E$ then

$$(E \otimes E)\{d\} = \{d^{E\otimes E}\}.$$

Therefore we can apply (2.F). Hence finite groups of odd order are solvable.

3. Proof of Theorem 1

We assume in this section that G is a finite group admitting a fixed-point-free automorphism f of order $2p$ for an odd prime p. Let A be a subgroup of G normalized by f. We use the following notations:

(i) $d = f^p$, $\quad k = f^2$.

(ii) $H = G\{f\}$, $D = d^G$.

(iii) $S_{q,A}$ is the Sylow q-subgroup of A normalized by f.

(iv) $\mathfrak{C}_A = \{C_A(d), (S_{2,A})^x;\ x \in A$ and $(S_A)^{xd} = (S_A)^x\}$.

Remark 3.1. The following properties of fixed-point-free automorphisms f are well known:

(′) If q is a prime then G contains a Sylow q-subgroup normalized by f.

(″) If U is a subgroup of G then f normalizes at most one element of U^G.

Lemma 3.2. $C_{(f^G)}(d) = [x \in f^G;\ x^p = d]$.

Proof. Let x be an element of f^G which commutes with d. Since C_H(d) is generated by the set $C_{(f^G)}(d)$, we get

$$C_{(f^G)}(d) = fC_G(d)$$

and

$$|f^{C_G(d)}| = |C_G(d)|.$$

Therefore $d = x^p$.

Lemma 3.3. *Let q be a prime such that*

$$C_{S_{q,G}}(d) = 1 = C_{S_{q,G}}(k).$$

Then G contains a normal q-complement.

Proof. We assume that G is a group of minimal order such that Lemma 3.3 is false. Then $q \neq 2$ and $\underset{\sim}{Q} = S_{q,G} \neq 1$. Since $\underset{\sim}{Q}$ is normalized by d, we get that $\underset{\sim}{Q}$ is abelian. The minimality of G and the lemma of Burnside imply that $C_G(\underset{\sim}{Q})$ is a proper subgroup of $N_G(\underset{\sim}{Q}) = G$. Hence there is a prime r such that $R = S_{r,G}$ induces nontrivial automorphisms of $\underset{\sim}{Q}$. The minimality of G implies $G = \underset{\sim}{Q}R$ and $\underset{\sim}{Q} = C_G(\underset{\sim}{Q})$. If $r = 2$ then k fixes no element of $\underset{\sim}{Q}R$ different from 1. If r is odd then R contains no element w such that

$$dwd = w^{-1} \neq 1,$$

since $\underset{\sim}{Q} = C_G(\underset{\sim}{Q})$. Hence G is nilpotent by the theorem of Thompson.

Lemma 3.4. *If G is solvable the following statements hold:*

(a) $G = (\{D\} \cap G)\mathfrak{C}_G$.

(b) *If R is a Sylow 2-subgroup of $\{D\}$ then*

$$\left[R: \left(\bigcap_{g \in \{D\}} R^g\right)\right] \leqslant 2.$$

(c) *D is a class of conjugate elements of $\{D, S_{2,G}\}$.*

(*d*) $D \cap S_{2,G}\{d\} = d^{S_{2,G}}$.

(e) *If S and S^* are different Sylow 2-subgroups of H then*

$$S \cap S^* \cap D = \phi.$$

(f) *$\mathfrak{C}_G$ is nilpotent.*

Proof. Let q be an odd prime. Set $\underset{\sim}{Q} = S_{q,G}$. Then

$$\underset{\sim}{Q}\{d\} = \{\underset{\sim}{Q}\{d\} \cap D\}C_{\underset{\sim}{Q}}(d),$$

since $\{d\}$ is a Sylow 2-subgroup of $\underset{\sim}{Q}\{d\}$. Therefore $\{D\}\mathfrak{C}_G$ contains for each prime r a Sylow r-subgroup of G and (a) is true. If X is a finite group let $\mathfrak{B}(X)$ be the intersection of the Sylow 2-subgroups of X. Let G be a group of minimal order such that (b) is false. The minimality of G implies that

$\{D\}$ contains exactly one minimal normal subgroup L of H and L is a $2'$-group. Let B be the subgroup of G satisfying

$$L \subseteq B \qquad \text{and} \qquad B/L = \mathfrak{B}(\{D\}/L).$$

Then $C_B(L)$ is a normal subgroup of H, since B is a normal subgroup of H. Since k operates fixed-point-free on $C_G(d)$, the theorem of Thompson implies that $C_G(d)$ is nilpotent. Because of Lemma 1.1 and the assumption that (b) is false, D contains an element $e \neq d$ which commutes with d. Hence $C_L(d) = C_L(e)$. Since d and e induce automorphism of order 2 of $L/C_L(d)$, we get that de is contained in $C_G(L)$. The minimality of G implies that de is contained in B. Hence $C_B(L) \neq L$ and B contains a nontrivial normal 2-subgroup. This contradiction proves (b).

Assume that d and e are different elements of D generating a 2-group. Let x be an element of H such that $e^x = d$. Then (b) implies that x normalizes $T = \{d, \mathfrak{B}(\{D\})\}$. Let w be an element of $N_G(T) \cap C_G(k)$. Then $dwd = w^{-1}$ and w is contained in $\{D\}$. Hence $w = 1$ and $N_G(T)$ is nilpotent. Therefore (c) and (d) are true. Since $\{D\}'$ is 2-closed by (b), different Sylow 2-subgroups of G$\{d\}$ containing d contain T. Since $N_G(T)$ is nilpotent (e) is a consequence of (d).

Because of (e), $\mathfrak{C}_G$ is 2-closed. Hence k fixes no element different from 1 of $\mathfrak{C}_G$. Hence (f) is true.

We assume now that G is a nonsolvable group of minimal order satisfying condition (I) or (II) of Theorem 1. We prove some consequences of this assumption which lead to a contradiction.

(1) *$G = G'$ is a minimal normal subgroup of G and proper subgroups of G normalized by f are solvable.*

Proof. Because of Remark 3.1, the subgroups of G normalized by f satisfy (I) or (II). If N is a minimal normal subgroup of H then $C_{G/N}(dN/N)$ contains a Sylow 2-subgroup of G/N if $C_G(d)$ contains a Sylow 2-subgroup of G. We assume that $C_G(d)$ is a 2-group. Let W be the subgroup of G satisfying

$$W \supseteq N \qquad \text{and} \qquad W/N = C_{G/N}(dN/N).$$

Then W is solvable and contains a Hall $2'$-subgroup L normalized by f. Hence $C_L(d) = 1$. Therefore we can assume that N is a 2-group and $W = NL$. Then $W\{d\}/N$ is 2-closed, which is impossible. Hence G/N satisfies (I) or (II).

(2) *$C_G(d)$ is a 2-group.*

Proof. Since $C_G(d)$ is nilpotent by the theorem of Thompson, it contains a unique maximal subgroup of U of odd order. We assume $U \neq 1$. Let e be an element of D which commutes with d. Then e is contained in $C_H(U)$ and

$C_H(e)$ normalizes U. Lemma 3.2 implies that d and e are conjugate in $N_H(U)$. Because of $U \neq 1$ and (1), $N_H(U)$ is solvable. Lemma 3.4 (d) and (e) imply that d and e are conjugate in each Sylow 2-subgroup of $N_H(U)$ containing d and e. If (2) is false we get from (II) that $C_G(d)$ contains a Sylow 2-subgroup of G. Hence Remark 3.1 implies $d = e$. Hence different elements of D do not commute. Since $C_{\{D\}}(d)$ is nilpotent we can apply Theorem 2, which leads to a contradiction.

(3) *G contains an abelian Hall $2'$-subgroup W.*

Proof. If q is an odd prime dividing the order of G, Lemma 3.3 and (2) imply that a Sylow q-subgroup Q of $C_G(k)$ it not trivial. Since $C_G(k)$ is abelian, $N_G(Q)$ contains $C_G(k)$ and a Sylow q-subgroup P of G. Since $N_G(Q)$ is solvable by (1), it contains an abelian Hall $2'$-subgroup W. Let r be a prime dividing the order of W. If R is a Sylow r-subgroup of W then $N_G(R)$ contains W and a Sylow r-subgroup of G. Since $N_G(R)$ is solvable R is a Sylow r-subgroup of G. Hence (3) is true.

Now, (3) implies that $G = WS_{2,G}$. The theorem of O. H. Kegel shows that G is solvable. This contradiction proves Theorem 1.

4. Proof of Theorem 3

We assume that G is a group of minimal order such that Theorem 3 is false.

(1) *G' is not nilpotent.*

Proof. If G' is nilpotent the minimality of G implies that G' has even order. Let S be a Sylow 2-subgroup of G'. Then

$$S = \{S\{d\}\}' \qquad \text{for} \qquad d \in D$$

and $S\{d\} \cap D$ is a class of conjugate involutions of $S\{d\}$. Since $S\{d\} \cap D$ generates a proper subgroup of $S\{d\}$, we get a contradiction.

(2) $Z(G') = 1$.

Proof. Assume $Z(G') \neq 1$. Then the order of $G/Z(G')$ is smaller than the order of G. Since $DZ(G')/Z(G')$ is a class of conjugate involutions generating $G/Z(G')$, the minimality of G and (1) imply that there is a prime p such that d normalizes $Z(G')P$ and $Z(G')Q$ for different Sylow p-subgroups of P and Q of G. Hence P and Q are not contained in G'. Then $G = G'\{d\}$ for $d \in D$ implies $p = 2$. The minimality of G implies that $Z(G')$ is a $2'$-group. Since

d normalizes a Sylow 2-subgroup S of $Z(G')P$ and Sylow 2-subgroup T of $Z(G')Q$ the hypothesis implies $S = T$. Hence

$$Z(G')P = Z(G')S = Z(G')Q.$$

This contradiction proves (2).

(3) *G is not solvable.*

Proof. If G is solvable G contains a minimal normal subgroup M which is a p-group. Hence G'/M is nilpotent. Therefore G contains exactly one minimal normal subgroup and G' is p-closed. Since G' is not nilpotent, there is a Sylow q-subgroup Q of G' that is not normal in G'. The minimality of G implies $G' = MQ$. Since $C_M(d)$ for $d \in N_D(Q)$ normalizes Q by assumption, $C_M(d)$ is a normal subgroup of G. Hence (2) implies $C_M(d) = 1$. The involution d induces an automorphism of Q/Q' which inverts each element of Q/Q'. Since q is odd, there is a set I of elements of Q such that

$$Q = \{I\} \qquad \text{and} \qquad did = i^{-1} \qquad \text{for} \qquad i \in I.$$

Hence d inverts each element of

$$M\{i\} \qquad \text{for} \qquad i \in I.$$

Therefore $M\{i\}$ is abelian and M is contained in $Z(G')$.

(4) *If d is an element of D then $C_G(d)$ is nilpotent.*

Proof. Let Q be a Sylow q-subgroup of $C_G(d)$ and P the Sylow q-subgroup of G normalized by d. Then Q normalizes P and Q is contained in P. Hence Q is normal in $C_G(d)$.

(5) *Different elements of D do not commute.*

Proof. Assume that d and e are elements of D which commute. Let Q be the Sylow q-subgroup of G normalized by d. Then e normalizes Q and (4) implies

$$C_Q(d) = C_Q(e)$$

if q is an odd prime. Then different elements of d^Q do not commute. Hence we can apply (2.C) for d^Q and $\{d^Q\}$. Therefore de centralizes for each odd prime a Sylow q-subgroup of G. Hence $[G : C_G(ed)]$ is a power of 2. If a Sylow 2-subgroup S of G contains ed then $(ed)^G \subseteq (ed)^S$. Therefore G contains a minimal normal subgroup M which is a 2-group. Hence G/M is solvable by the minimality of G.

Because of (4) and (5), the hypothesis of Theorem 2 is verified. Hence G is solvable.

Remark 4.1. Let F be the Frobenius group of order $5^2 \cdot 2^3 \cdot 3$ with quaternion Sylow 2-subgroup and $D \neq 1$ a class of conjugate 3-elements. Then

$$F = \{D\} = F'\{d\} \qquad \text{for} \qquad d \in D$$

and d normalizes exactly one Sylow p-subgroup of F for $p = 2, 3, 5$. But F' is not nilpotent.

5. Proof of Theorem 4

LEMMA 5.1. *Let p be a prime, N a proper normal subgroup of the solvable group G, and d a p-element of G such that*

(a) $G = N\{d\}$ *and* $N \cap \{d\} = 1$;

(b) *if S and R are different Sylow p-subgroups of G then $S \cap R$ is contained in N.*

Then N is p-closed.

Proof. Let G be group of minimal order such that Lemma 5.1 is false. Because of (a), d has order p. Let M be a minimal normal subgroup of G which is contained in N. Then (a) and (b) are true for G/M. Hence N/M is p-closed by the minimality of G, and N contains exactly one minimal normal subgroup of G. Since M is not a p-group, we get for a Sylow p-subgroup P of N

$$G = MP\{d\}.$$

Hence $C_P(M) = 1$. If $C_G(M) \neq M$ then

$$G = C_G(M)P,$$

and $Z(G)$ contains an element of order p which is not contained in N; hence (b) implies in this case that G is nilpotent. Hence

$$C_G(M) = M. \tag{$^+$}$$

Let z be an element of $Z(P)$ of order p. We can assume that z commutes with d. The minimality of G and $(+)$ imply

$$G = M\{z\}\{d\}.$$

Since $\{z^G\}$ and $\{d^G\}$ are nontrivial normal subgroups of G we get

$$M \cap \{z^M\} = M \cap \{z^G\} = M = M \cap \{d^G\} = M \cap \{d^M\}.$$

Hence

$$1 = C_M(z) = C_M(d).$$

Since $\{z, d\}$ is elementary abelian of order p^2, there is an element w in $\{z, d\}$ which is not contained in N such that $C_M(w) \neq 1$ since G is not a Frobeniusgroup (see [*1*, Satz 4.1 (5)]). Since M is a minimal normal subgroup of G we get that w is contained in $Z(G)$. This is impossible because of (b).

We assume now that G is a group of minimal order such that Theorem 4 is false. Then d is an involution.

(1) *$N = G' = G''$ is a minimal normal subgroup of G.*

Proof. Let M be a minimal normal subgroup of G contained in N. Since $M\{d\}$ satisfies the hypothesis, the minimality of G implies that M is 2-closed if $M \neq N$. Since M is contained in N, we get that G/M satisfies the hypothesis, too. The theorem of Feit and Thompson implies that G is solvable. Hence N is 2-closed by Lemma 5.1. This contradiction proves (1).

Let S be a Sylow 2-subgroup of G containing d, set

$$T = [x \in G;\, x^2 = 1 \text{ and } x \notin N].$$

(2) *If U is a proper subgroup of G with*

$$|\, U \cap T \cap S \,| \geqslant 2$$

then U normalizes $S \cap N$.

Proof. We can assume that U is a maximal subgroup of G. Let Q be a Sylow 2-subgroup of U. We can assume that Q contains d. Then

$$N_U(Q) \subseteq N_G(S),$$

since d is contained in S and in no other Sylow 2-subgroup of G. The minimality of G implies that $N \cap U$ is 2-closed. Let R be Sylow 2-subgroup of $N \cap U$. Since U contains different elements of $T \cap S$, we get $R \neq 1$. Since U is maximal we get from (1)

$$N_G(R) = U.$$

Now, $R = Q \cap N$ implies that R is a Sylow 2-subgroup of N. Hence

$$R = S \cap N.$$

(3) *If x is an involution of G then x is contained in T.*

Proof. Let x be an involution of G which is not contained in T. Since T is a normal complex of G, we get for an element t of $T \cap S$ that xt has even order. Since $\{x, t\}$ is a dihedral group, there is exactly one involution z_t in $Z(\{x, t\})$ different from x and t. Then z_t is contained in S. Assume

$$C_{T \cap S}(z_t) = t \qquad \text{for each} \qquad t \text{ in } T \cap S.$$

Then z_t is not contained in N and $Z(S) \cap \cap N = 1$. Therefore N has odd order. This contradiction proves that $T \cap S$ contains an element w such that

$$C_{T \cap S}(z_w) \neq w.$$

Hence (2) proves

$$x \in C_G(z_w) \subseteq N_G(S \cap N).$$

Therefore

$$\{x^G\} \subseteq N_G(S \cap N).$$

Since S normalizes $N \cap S$ we get together with (1)

$$N_G(S \cap N) \supseteq \{S, x^G\} \supseteq \{S, G'\} = G.$$

Hence $S \cap N$ is a normal 2-subgroup of G. This contradiction proves that x is contained in T.

But (3) implies that N has odd order, hence N is 2-closed.

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